Zero-sum game

Zero-sum game

We say that a game is a zero-sum game if the sum of payoffs of all players is always zero.

Product game

Player I chooses a number from ‘2’ or ‘-1’ and player II chooses a number from ‘1’ or ‘-2’ simultaneously. Then player II gives $$p$ to player I where p is the product of the two numbers. (Player 1 pays player 2 if p is negative.)

Product game

This is a zero-sum game.

		Player II (Column Player)	
		1	-2
Player I (Row Player)	2	(2,-2)	(-4,4)
	-1	(-1,1)	(2,-2)

Product game

We may use two matrices *A* and *B* to represent the payoffs of players I and II respectively.

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

Note that B is just the negative of A.

Game matrix

Therefore we may use one single matrix to represent a zero-sum game. Suppose the payoffs of player I is represented by the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Then the payoffs of player II is just

$$B = -A$$

Saddle point

We say that the entry a_{ij} is a saddle point of

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

if a_{ij} is the largest number in its column and the smallest number in its row, in other words

1. $a_{ij} \geq a_{kj} \quad \text{for any } 1 \leq k \leq m.$
2. $a_{ij} \leq a_{il} \quad \text{for any } 1 \leq l \leq n.$

Saddle point

Examples

1.

$$\begin{pmatrix} 3 & -2 \\ 2 & \boxed{1} \end{pmatrix}$$

2.

$$\begin{pmatrix} 0 & -2 & -4 \\ \boxed{1} & 3 & 2 \\ -1 & 0 & -2 \end{pmatrix}$$

3.

$$\begin{pmatrix} 1 & -2 & -4 & 2 \\ -2 & 4 & -5 & 2 \\ 3 & 1 & \boxed{-2} & 0 \\ -4 & 0 & -3 & 3 \end{pmatrix}$$

Saddle point

Suppose a_{ij} is a saddle point of

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

1. **If player I uses the *i*-th strategy, then no matter how player II plays**

$$\text{payoff of I} \geq a_{ij}$$

2. **If player II uses the *j*-th strategy, then no matter how player I plays**

$$\text{payoff of I} \leq a_{ij}$$

Value of game

We say that the **value of game** is v if

1. there exists a strategy of I, called the **maximin strategy**, such that no matter how II plays

$$\boxed{\text{payoff of I} \geq v}$$

2. there exists a strategy of II, called the **minimax strategy**, such that no matter how I plays

$$\boxed{\text{payoff of I} \leq v}$$

Value of game

Suppose *A* has a saddle point a_{ij}. Then the value of the game is

$$\boxed{v = a_{ij}}$$

Uniqueness of value

There can be more than one saddle point. However, the value of any two saddle points must be the same.

***Theorem*. Suppose a_{ij} and a_{kl} are saddle points. Then**

$$\boxed{a_{ij} = a_{kl}}$$

***Proof*. Using alternatively that a_{ij} and a_{kl} are saddle points, we have**

$$a_{ij} \le a_{il} \le a_{kl} \le a_{kj} \le a_{ij}$$

Therefore

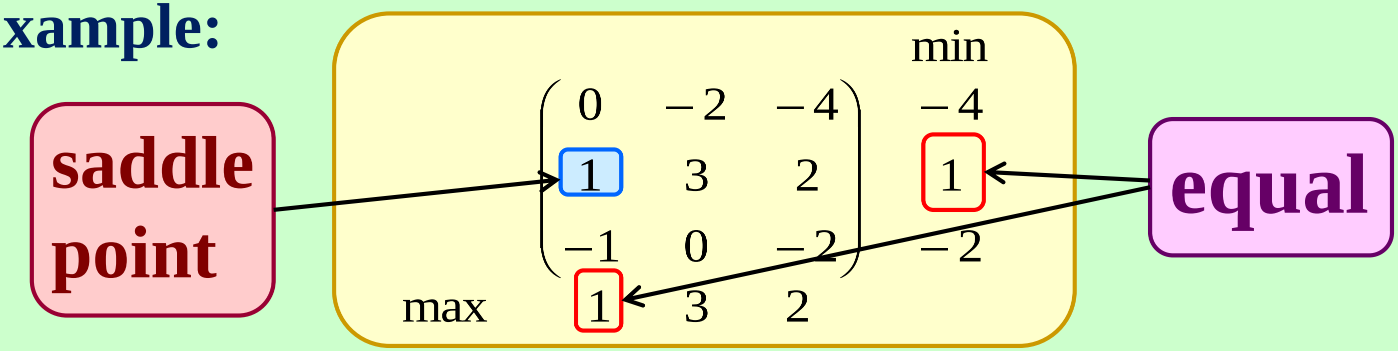
$$a_{ij} = a_{kl}$$

$$\begin{pmatrix} & \vdots & & \vdots & \\ \cdots & a_{ij} & \cdots & a_{il} & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & a_{kj} & \cdots & a_{kl} & \cdots \\ & \vdots & & \vdots & \end{pmatrix}$$

Finding saddle point

1. **Write down the minimum entry of each row.**
2. **Write down the maximum entry of each column.**
3. **If the maximum of the row minima (maximin) and the minimum of the column maxima (minimax) are equal, then there is a saddle point at the corresponding entry.**

Example:



$$\begin{array}{rcc} & & \text{min}\\ & \left(\begin{array}{ccc}0&-2&-4\\ \boxed{1}&3&2\\ -1&0&-2\end{array}\right) & \begin{array}{c}-4\\ \boxed{1}\\ -2\end{array}\\ \text{max} & \begin{array}{ccc}\boxed{1}&3&2\end{array} & \end{array}$$

saddle point

equal

Finding saddle point

Example:

$$\begin{array}{lcc}
 & & \text{min} \\
 & \begin{pmatrix} 1 & -2 & -4 & 2 \\ -2 & 4 & -5 & 2 \\ 3 & 1 & \boxed{-2} & 0 \\ -4 & 0 & -3 & 3 \end{pmatrix} & \begin{array}{c} -4 \\ -5 \\ \boxed{-2} \\ -4 \end{array} \\
\text{max} & \begin{array}{cccc} 3 & 4 & \boxed{-2} & 3 \end{array} &
\end{array}$$

There is a saddle point at the 3,3 entry.

The value of the game is -2.

Finding saddle point

Example:

$$\begin{array}{cccccc}
 & & & & & \text{min} \\
 & 2 & -2 & 3 & 5 & \boxed{-2} \\
 & 7 & 1 & -4 & 3 & -4 \\
 & -2 & -3 & 0 & 2 & -3 \\
 & 1 & 0 & 4 & -2 & \boxed{-2} \\
\text{max} & 7 & \boxed{1} & 4 & 5 &
\end{array}$$

The maximin and minimax are not equal.

The game has no saddle point.

Pure and mixed strategies

Pure strategy

Constantly using one strategy.

Mixed strategy

Using different strategies according to certain probabilities.

Pure and mixed strategies

Suppose I has *m* strategies and II has *n* strategies (the game is represented by an *m* by *n* matrix). Then a mixed strategy of I is a vector

$$\mathbf{p} = (p_1, p_2, \cdots, p_m)$$

where

1. $0 \le p_i \le 1 \text{ for any } i = 1, 2, \cdots, m$

2. $p_1 + p_2 + \cdots + p_m = 1$

Here p_i is the probability that I uses the *i*-th strategy.

Pure and mixed strategies

Similarly a mixed strategy of II is

$$\boxed{\mathbf{q} = (q_1, q_2, \cdots, q_n)}$$

Note that a pure strategy is also a mixed strategy with one of p_i's equal to 1 and all other p_i's equal to 0.

Pure and mixed strategies

Recall that if I uses $\mathbf{p}=\left(p_1,p_2,\cdots,p_m\right)$ and II uses $\mathbf{q}=\left(q_1,q_2,\cdots,q_n\right)$, then the expected payoff of I is

$$E\left(P_A\right)=a_{11}p_1q_1+a_{12}p_1q_2+\cdots+a_{kl}p_kq_l+\cdots+a_{mn}x_my_n$$

$$=\begin{pmatrix}p_1 & p_2 & \cdots & p_m\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}q_1\\ q_2\\ \vdots\\ q_n\end{pmatrix}$$

$$=\mathbf{p}A\mathbf{q}^T$$

Minimax theorem

For any finite two-person zero-sum game, the **value of the game** exists. In other words, there exists real number *v* such that

1. there exists mixed strategy of I, which is called **maximin strategy**, such that the payoff of I is at least *v* no matter how II plays,
2. there exists mixed strategy of II, which is called **minimax strategy**, such that the payoff of I is at most *v* no matter how I plays.

2-by-2 game

To solve 2-by-2 game

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. **Find if there is a saddle point.**
2. **If there is no saddle point, suppose I uses**

$$\mathbf{p} = (p, 1 - p), \quad 0 \le p \le 1$$

2-by-2 game

$$\mathbf{p}A = (p, 1-p)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (ap + (1-p)c, bp + (1-p)d)$$

$$= ((a-c)p + c, (b-d)p + d)$$

That means the payoff of I is

1. **$(a - c)p + c$ if II uses 1st strategy.**
2. **$(b - d)p + d$ if II uses 2nd strategy.**

2-by-2 game

Now if

$$(a-c)p+c=(b-d)p+d$$

$$(a-b+d-c)p=d-c$$

$$p=\frac{d-c}{a-b+d-c}$$

then the payoff of I will be fixed no matter how II plays.

2-by-2 game

In this case the payoff of I is always

$$(a-c)p+c=(a-c)\left(\frac{d-c}{a-b+d-c}\right)+c$$

$$=\frac{\left(ad-ac-cd+c^{2}\right)+\left(ac-bc+cd-c^{2}\right)}{a-b+d-c}$$

$$=\frac{ad-bc}{a-b+d-c}$$

no matter how II plays.

2-by-2 game

Similarly suppose II uses

$$\mathrm{q} = (q, 1-q), \quad 0 \le q \le 1$$

Then the payoff of I is given by the vector

$$A\mathrm{q}^{\mathrm{T}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= \begin{pmatrix} aq+b(1-q) \\ cq+d(1-q) \end{pmatrix}$$

$$= \begin{pmatrix} (a-b)q+b \\ (c-d)q+d \end{pmatrix}$$

payoffs of I if I uses 1st strategy

payoffs of I if I uses 2nd strategy

2-by-2 game

When

$$\begin{aligned}(a-b)q+b&=(c-d)q+d\\(a-b+d-c)q&=d-b\\q&=\frac{d-b}{a-b+d-c}\end{aligned}$$

no matter how I plays, the payoff of I will be

$$\begin{aligned}(a-b)q+b&=(a-b)\left(\frac{d-b}{a-b+d-c}\right)+c\\&=\frac{ad-bc}{a-b+d-c}\end{aligned}$$

Solution of 2-by-2 game

If the 2-by-2 game $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **has no saddle point, then its value is**

$$v=\frac{ad-bc}{a-b+d-c}$$

1. **maximin strategy for I:** $\mathbf{p}=\left(\frac{d-c}{a-b+d-c},\frac{a-b}{a-b+d-c}\right)$

The payoff of I is equal to *v* no matter how II plays.

2. **minimax strategy for II:** $\mathbf{q}=\left(\frac{d-b}{a-b+d-c},\frac{a-c}{a-b+d-c}\right)$

The payoff of I is equal to *v* no matter how I plays.

Product game

Player I chooses two numbers ‘2’ or ‘-1’ and player II chooses two numbers ‘1’ or ‘-2’ simultaneously. Then player II gives $*p* to player I where *p* is the product of the two numbers. (Player I pays player II if *p* is negative.)

Game matrix

Strategy of player I: “2”, “-1”
Strategy of player II: “1”, “-2”

The payoffs of I is given by the game matrix

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

The payoff matrix of II is

$$B = -A = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

Solution

The matrix *A* has no saddle point.

Suppose I uses

$$\mathbf{p} = (p, 1-p), \quad 0 \le p \le 1$$

The payoff of I is given by the row vector

$$\mathbf{p}A = \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2p-(1-p) & -4p+2(1-p) \end{pmatrix}$$

$$= \begin{pmatrix} 3p-1 & -6p+2 \end{pmatrix}$$

Solution

The payoff of I will be fixed no matter how II plays if

$$3p-1=-6p+2$$
$$9p=3$$
$$p=\frac{1}{3}$$

Now

$$\begin{pmatrix}\frac{1}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}=\begin{pmatrix}0 & 0\end{pmatrix}$$

Hence if I uses $\boxed{\left(\frac{1}{3},\frac{2}{3}\right)}$, his payoff will be $\boxed{0}$ no matter how II plays.

Solution

Similarly suppose II uses

$$\mathrm{q} = (q, 1-q), \quad 0 \le q \le 1$$

The payoff of II is given by the column vector

$$A\mathrm{q}^{\mathrm{T}} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= \begin{pmatrix} 2q - 4(1-q) \\ -q + 2(1-q) \end{pmatrix}$$

$$= \begin{pmatrix} 6q - 4 \\ -3q + 2 \end{pmatrix}$$

Solution

The payoff of I will be fixed no matter how I plays if

$$\begin{aligned} 6q-4 &= -3q+2 \\ 9q &= 6 \\ q &= \frac{2}{3} \end{aligned}$$

Now

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence if II uses $\boxed{\left(2/3, 1/3\right)}$, the payoff of I will be $\boxed{0}$ no matter how I plays.

Solution

Therefore the value of the game matrix

$$\boxed{A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}}$$

is

$$\boxed{v = 0}$$

1. When I uses $\boxed{\left(1/3, 2/3\right)}$, the payoff of I is $\boxed{0}$ no matter how II plays.
2. When II uses $\boxed{\left(2/3, 1/3\right)}$, the payoff of I is $\boxed{0}$ no matter how I plays.

Using formulas

One may solve the game

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

using the formulas derived previously.

The value of the game is

$$v = \frac{ad - bc}{a - b + d - c} = \frac{2(2) - (-4)(-1)}{2 - (-4) + 2 - (-1)} = 0$$

Using formulas

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

1. The maximin strategy for I is

$$\mathbf{p} = \left(\frac{d-c}{a-b+d-c}, \frac{a-b}{a-b+d-c} \right)$$

$$= \left(\frac{2-(-1)}{2-(-4)+2-(-1)}, \frac{2-(-4)}{2-(-4)+2-(-1)} \right)$$

$$= \left(\frac{1}{3}, \frac{2}{3} \right)$$

Using formulas

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

2. The minimax strategy for II is

$$\mathbf{q} = \left(\frac{d-b}{a-b+d-c}, \frac{a-c}{a-b+d-c} \right)$$

$$= \left(\frac{2-(-4)}{2-(-4)+2-(-1)}, \frac{2-(-1)}{2-(-4)+2-(-1)} \right)$$

$$= \left(\frac{2}{3}, \frac{1}{3} \right)$$

Oddment method

$$\begin{array}{ll} A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} & \boxed{\begin{matrix} 6 \\ -3 \end{matrix}} \\ \qquad \boxed{\begin{matrix} 3 & -6 \end{matrix}} & \end{array}$$

difference of entries in rows

difference of entries in columns

Oddment method

$$A = \begin{array}{cc} \left(\begin{array}{cc} 2 & -4 \\ -1 & 2 \end{array}\right) & \begin{array}{c} 6 \\ -3 \end{array} \\ \begin{array}{cc} 3 & -6 \end{array} & \end{array}$$

opposite signs

opposite signs

No saddle point if both are of opposite signs.

Oddment method

$$A=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\begin{array}{ccc}6 & & 3\\ & \times & \\ -3 & & 6\end{array}$$

$$\begin{array}{cc}3 & -6\\ \multicolumn{2}{c}{\times}\\ 6 & 3\end{array}$$

Interchange and forget the signs

Oddment method

$$A=\begin{pmatrix}2 & -4\\ -1 & 2\end{pmatrix}\begin{array}{ccc}6 & & 3\\ & \times & \\ -3 & & 6\end{array}\quad\boxed{\begin{array}{c}1/3\\ 2/3\end{array}}$$

$$\begin{array}{cc}3 & -6\\ \multicolumn{2}{c}{\times}\\ 6 & 3\\ \hline 2/3 & 1/3\end{array}$$

Ratios of the numbers give the strategies.

Oddment method

$$A=\begin{pmatrix}2&-4\\-1&2\end{pmatrix}\begin{matrix}6\\-3\end{matrix}\ \times\ \begin{matrix}3\\6\end{matrix}\ \boxed{\begin{matrix}1/3\\2/3\end{matrix}}$$

$$\begin{array}{ccc}3 & & -6\\ & \times & \\ 6 & & 3\\ 2/3 & & 1/3\end{array}$$

maximin strategy for I

$$\mathbf{p}=\left(\frac{1}{3},\frac{2}{3}\right)$$

minimax strategy for II

$$\mathbf{q}=\left(\frac{2}{3},\frac{1}{3}\right)$$

Example

Solve the 2-by-2 game

$$A = \begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix}$$

Example

$$\begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix} \begin{matrix} 8 \\ -2 \end{matrix} \times \begin{matrix} 2 & 0.2 \\ 8 & 0.8 \end{matrix}$$

$$3 \quad -7$$

$$\times$$

$$7 \quad 3$$

$$0.7 \quad 0.3$$

Example

maximin strategy for I:

$$\mathbf{p} = (0.2, 0.8)$$

minimax strategy for II:

$$\mathbf{q} = (0.7, 0.3)$$

value of the game:

$$v = 2.6$$

$$\begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix} \begin{matrix} 8 & & 2 \\ & \times & \\ -2 & & 8 \end{matrix} \quad \boxed{\begin{matrix} 0.2 \\ 0.8 \end{matrix}}$$

$$\begin{matrix} 3 & -7 \\ & \times \\ 7 & 3 \end{matrix}$$

$$\boxed{0.7 \quad 0.3}$$

$$\mathbf{p}A = \begin{pmatrix} 0.2 & 0.8 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2.6 & 2.6 \end{pmatrix}$$

$$A\mathbf{q}^{\mathrm{T}} = \begin{pmatrix} 5 & -3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.6 \\ 2.6 \end{pmatrix}$$

Modified rock-paper-scissors

Player I strategies: Rock (R), Paper (P)

Player II strategies: Rock (R), Scissors (S)

		Player II (Column Player)	
		R	S
Player I (Row Player)	R	(0,0)	(1,-1)
	P	(1,-1)	(-1,1)

Modified rock-paper-scissors

The payoff matrix of Player I is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

The payoff of Player II is just its negative

$$B = -A = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

Modified rock-paper-scissors

Suppose Player II uses strategy (0.2,0.8). Calculate

$$A\mathbf{q}^T = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.8 \\ -0.6 \end{pmatrix}$$

The expected payoff of Player I is 0.8 if he uses rock(R).

The expected payoff of Player I is -0.6 if he uses rock(P).

Modified rock-paper-scissors

Suppose Player I uses strategy (0.4,0.6) and Player II uses strategy (0.2,0.8). The expected payoff of Player I is

$$\begin{aligned}\mathbf{p}A\mathbf{q}^T &= \begin{pmatrix}0.4 & 0.6\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0.2\\ 0.8\end{pmatrix}\\ &= \begin{pmatrix}0.4 & 0.6\end{pmatrix}\begin{pmatrix}0.8\\ -0.6\end{pmatrix}\\ &= -0.04\end{aligned}$$

The expected payoff of Player II is 0.04.

Modified rock-paper-scissors

Let $\mathbf{p} = (p, 1-p)$

$$\begin{aligned}\mathbf{p}A &= \begin{pmatrix} p & 1-p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1-p & p-(1-p) \end{pmatrix} \\ &= \begin{pmatrix} 1-p & 2p-1 \end{pmatrix}\end{aligned}$$

Equating

$$\begin{aligned} 1-p &= 2p-1 \\ 3p &= 2 \\ p &= \frac{2}{3} \end{aligned}$$

Therefore the maximin strategy for I is $\mathbf{p} = \left(\frac{2}{3}, \frac{1}{3}\right)$

Modified rock-paper-scissors

Let $\mathbf{q} = (q, 1-q)$

$$A\mathbf{q}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= \begin{pmatrix} 1-q \\ 2q-1 \end{pmatrix}$$

Equating

$$1-q = 2q-1$$

$$3q = 2$$

$$q = \frac{2}{3}$$

Therefore the minimax strategy for II is $\mathbf{q} = \left(\frac{2}{3}, \frac{1}{3}\right)$

Modified rock-paper-scissors

Now

$$\mathbf{p}A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$A\mathbf{q}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$$

The value of the game is

$$v = \frac{1}{3}$$

Modified rock-paper-scissors

We may also use the formulas. First we calculate

$$a-b+d-c=0-1+\left(-1\right)-1=-3$$

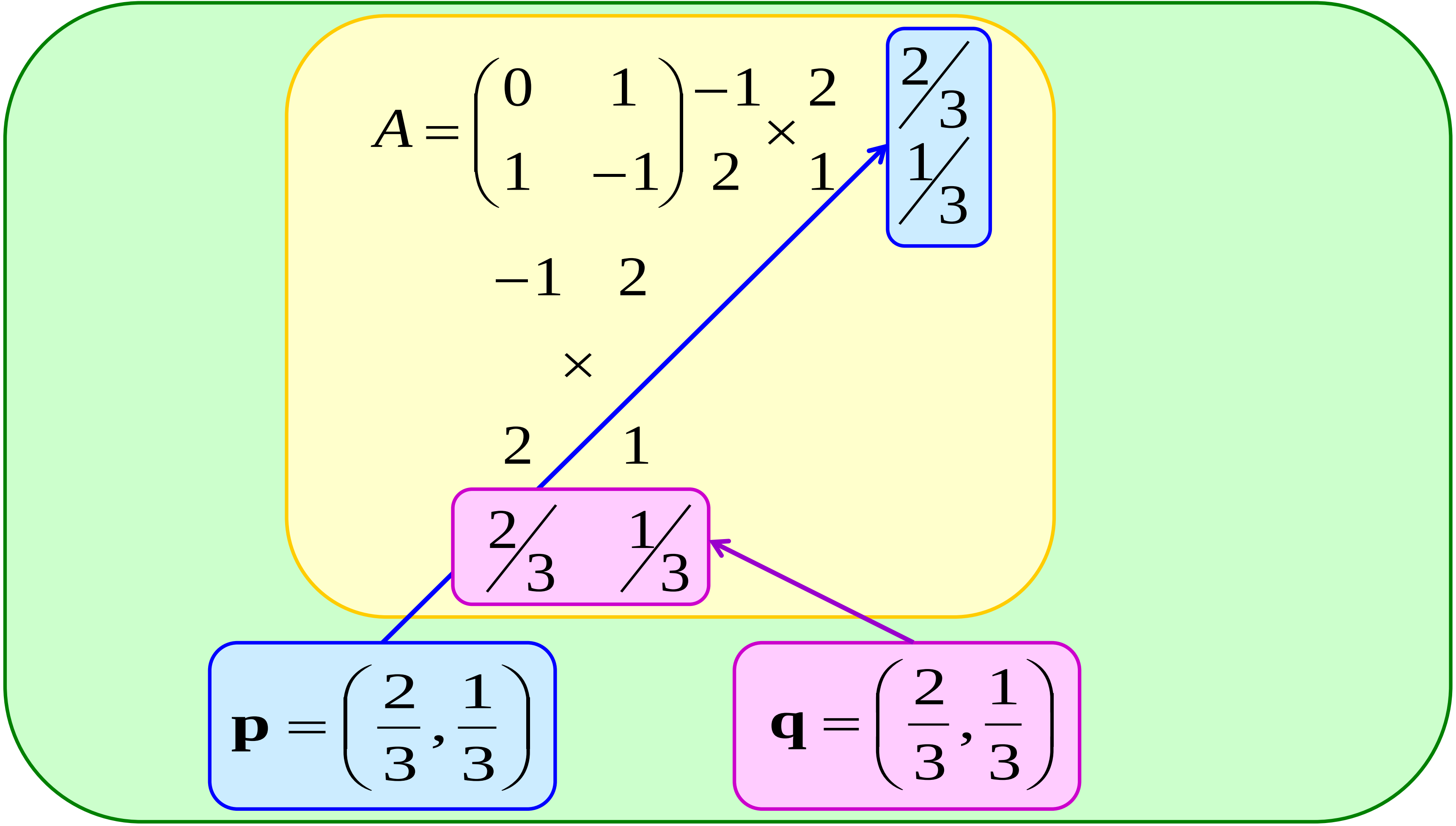
Then

$$\mathbf{p}=\left(\frac{d-c}{a-b+d-c},\frac{a-b}{a-b+d-c}\right)=\left(\frac{-1-1}{-3},\frac{0-1}{-3}\right)=\left(\frac{2}{3},\frac{1}{3}\right)$$

$$\mathbf{q}=\left(\frac{d-b}{a-b+d-c},\frac{a-c}{a-b+d-c}\right)=\left(\frac{-1-1}{-3},\frac{0-1}{-3}\right)=\left(\frac{2}{3},\frac{1}{3}\right)$$

$$v=\frac{ad-bc}{a-b+d-c}=\frac{0\left(-1\right)-\left(-1\right)\left(-1\right)}{-3}=\frac{1}{3}$$

Modified rock-paper-scissors



$$A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{array}{ccc} -1 & & 2 \\ & \times & \\ 2 & & 1 \end{array} \quad \boxed{\begin{array}{c} {}^{2}\!/_{3} \\ {}^{1}\!/_{3} \end{array}}$$

$$\begin{array}{ccc} -1 & & 2 \\ & \times & \\ 2 & & 1 \end{array}$$

$$\boxed{{}^{2}\!/_{3} \quad {}^{1}\!/_{3}}$$

$$\boxed{\mathbf{p} = \left(\frac{2}{3}, \frac{1}{3}\right)} \qquad \boxed{\mathbf{q} = \left(\frac{2}{3}, \frac{1}{3}\right)}$$

2-by-*n* game

Consider the 2-by-*n* game

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$

Assume that *A* has no saddle point.

Suppose I uses

$$\mathbf{p} = (p, 1 - p), \quad 0 \le p \le 1$$

2-by-*n* game

Consider

$$\mathbf{p}A = \begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}$$
$$= \left(a_{11}p + a_{21}(1-p) \quad \cdots \quad a_{1n}p + a_{2n}(1-p) \right)$$

Then II would use the *k*-th strategy so that

$$\boxed{a_{1k}p + a_{2k}(1-p)}$$

is minimum among the entries in $\mathbf{p}A$.

2-by-*n* game

Then player I may guarantee that his payoff is at least

$$\min_{1\le k\le n}\left\{a_{1k}p+a_{2k}(1-p)\right\}$$

The above expression simply means the minimum value of the entries in $\mathbf{p}A$.

The maximin strategy of I will be corresponding to a value of *p* so that the above expression is maximum.

2-by-*n* game

The problem can be solved graphically.

1. For each *j* = 1,2,…,*n*, draw the straight line

$$y = a_{1j}x + a_{2j}(1-x), \quad 0 \le x \le 1$$

2. Draw the polygonal curve for

$$y = \min_{1\le k\le n}\{a_{1k}p + a_{2k}(1-p)\}$$

3. Find *p* for which the above expression attains it maximum.

4. Find the maximin strategy for I, the minimax strategy for II and the value of the game.

Example 1

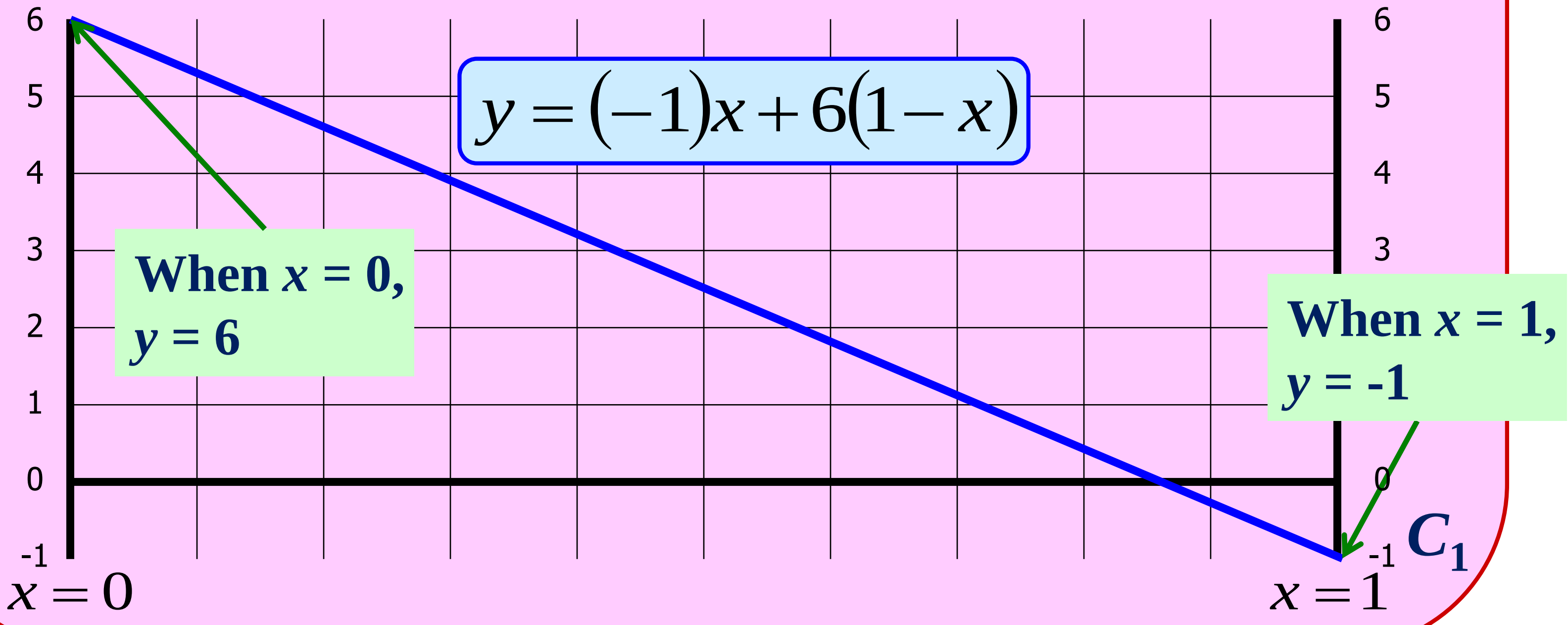
Solve the game matrix

$$A = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

Example 1

$$A=\begin{pmatrix}\boxed{\begin{matrix}-1\\ 6\end{matrix}} & \begin{matrix}5\\ -1\end{matrix} & \begin{matrix}3\\ 0\end{matrix} & \begin{matrix}2\\ 4\end{matrix}\end{pmatrix}$$

1. Draw a line for each column.



$$\boxed{y=(-1)x+6(1-x)}$$

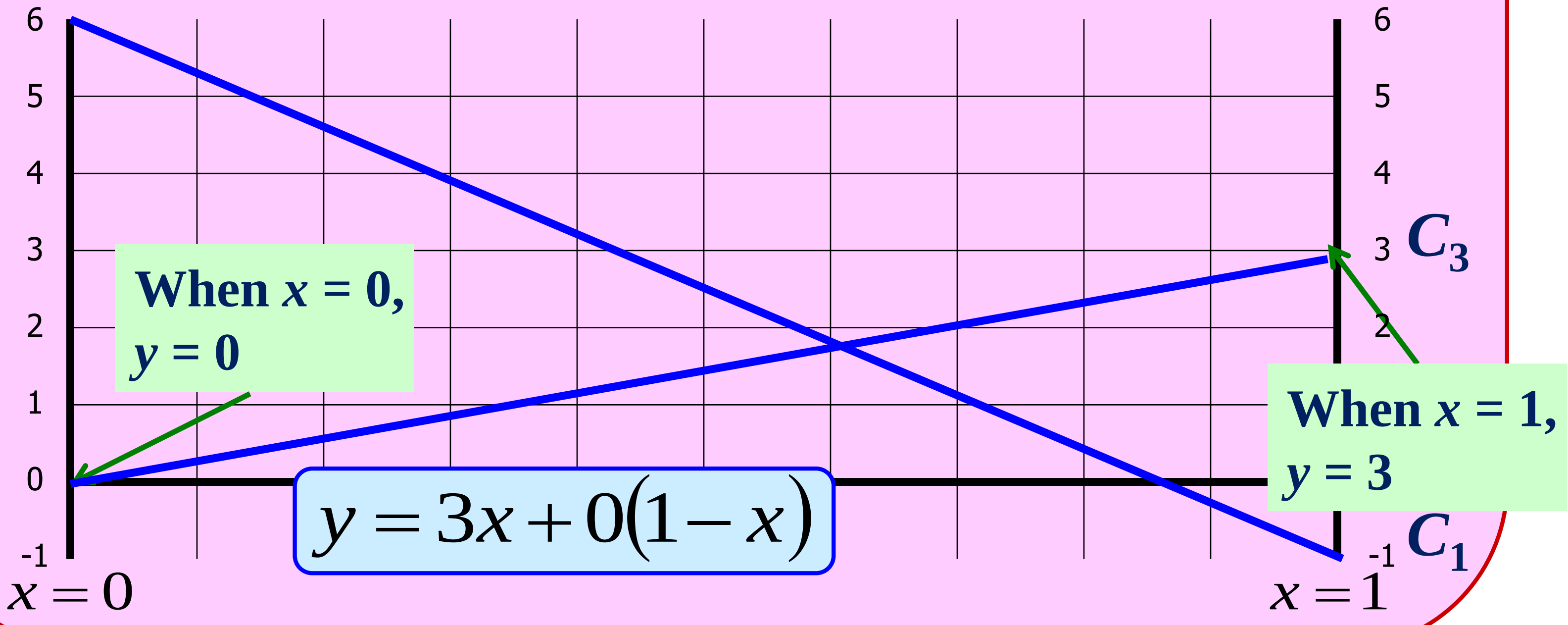
When x = 0, y = 6

When x = 1, y = -1

Example 1

$$A=\begin{pmatrix} -1 & 5 & \boxed{\begin{matrix}3\\0\end{matrix}} & \begin{matrix}2\\4\end{matrix} \end{pmatrix}$$

1. Draw a line for each column.



When $x = 0$, $y = 0$

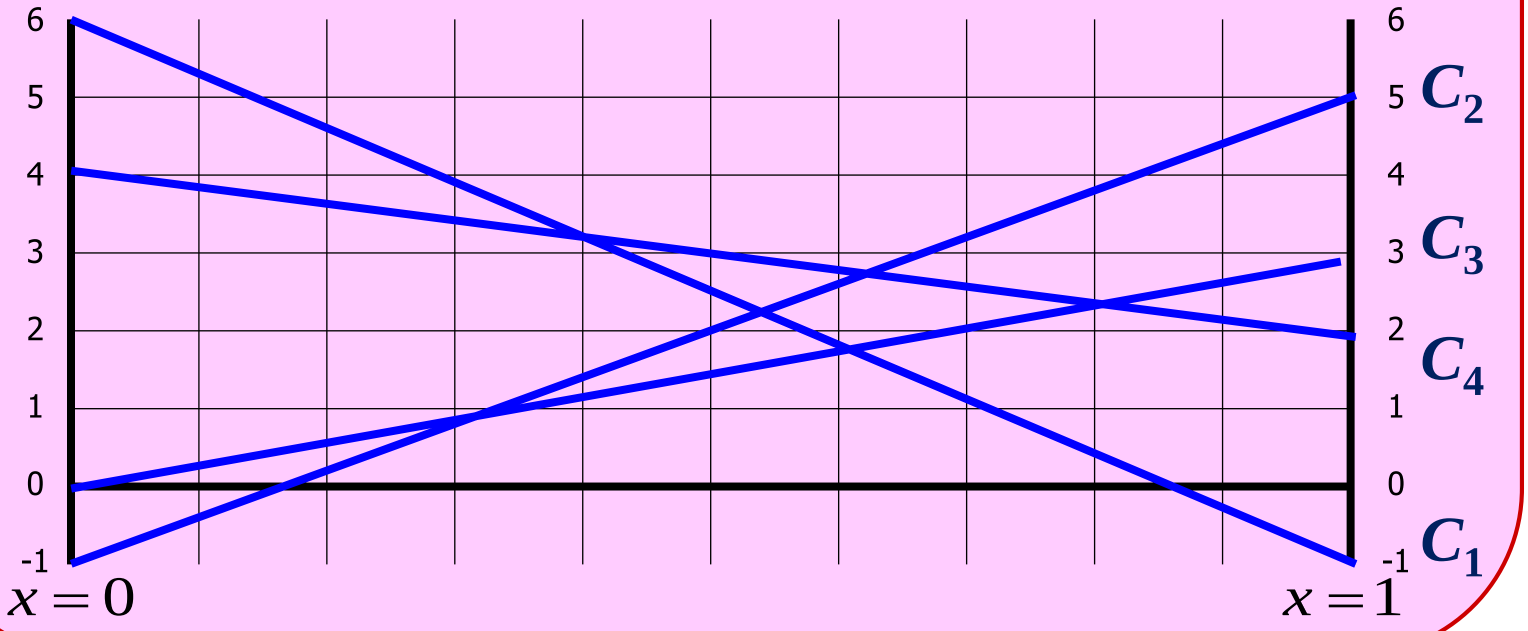
$$y=3x+0(1-x)$$

When $x = 1$, $y = 3$

Example 1

$$A = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

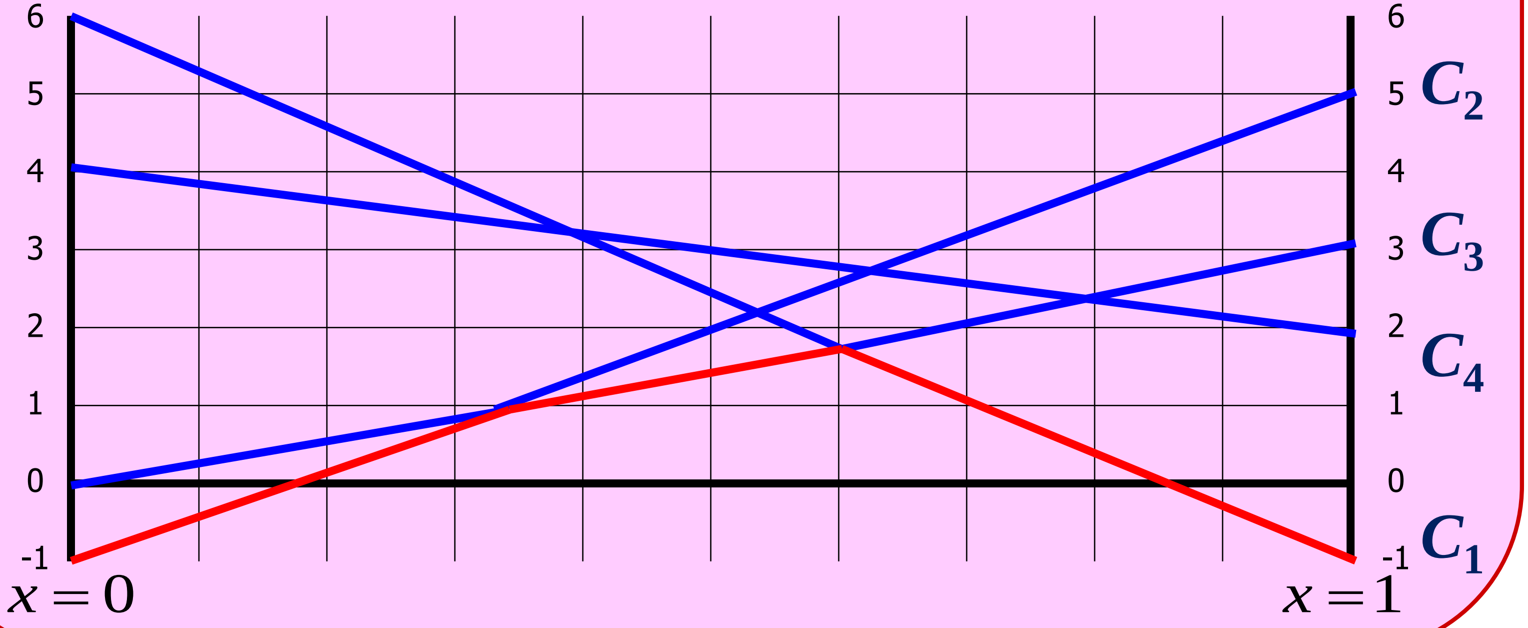
1. **Draw a line for each column.**



Example 1

$$A = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

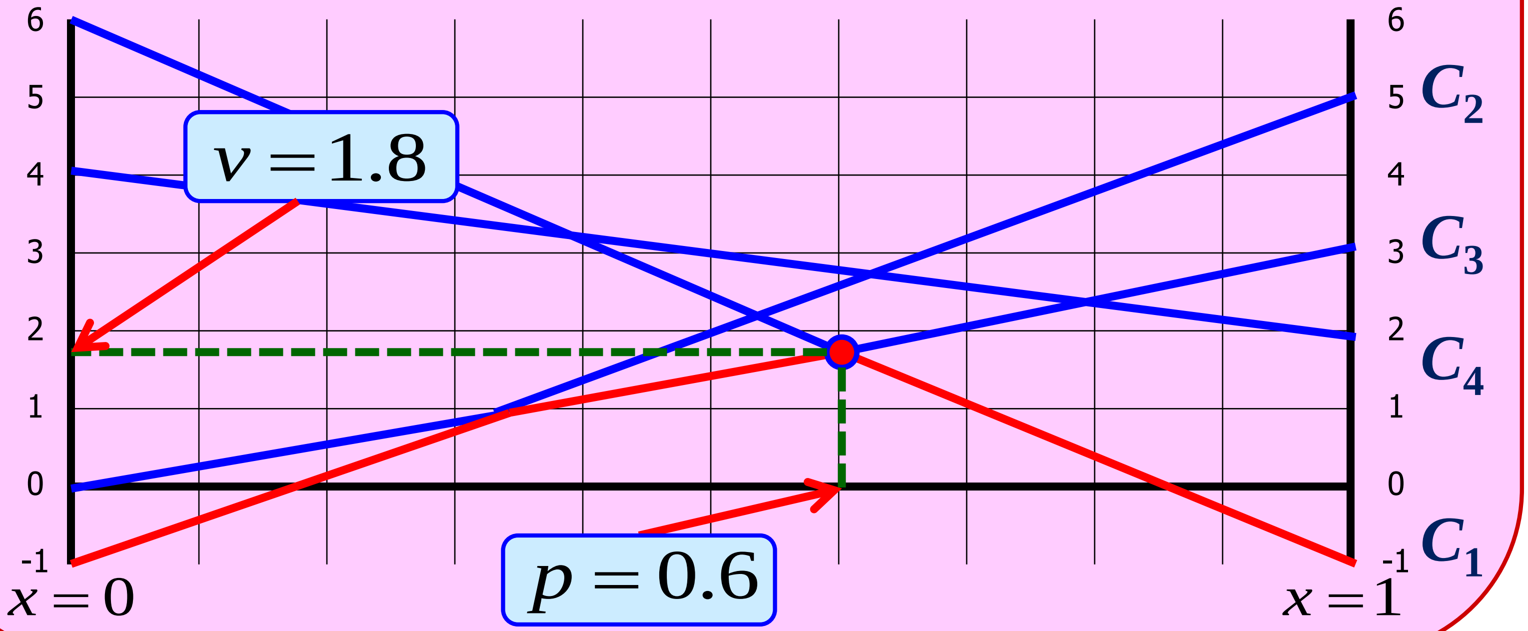
2. **Draw the polygonal curve for minimum *y*.**



Example 1

$$A = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

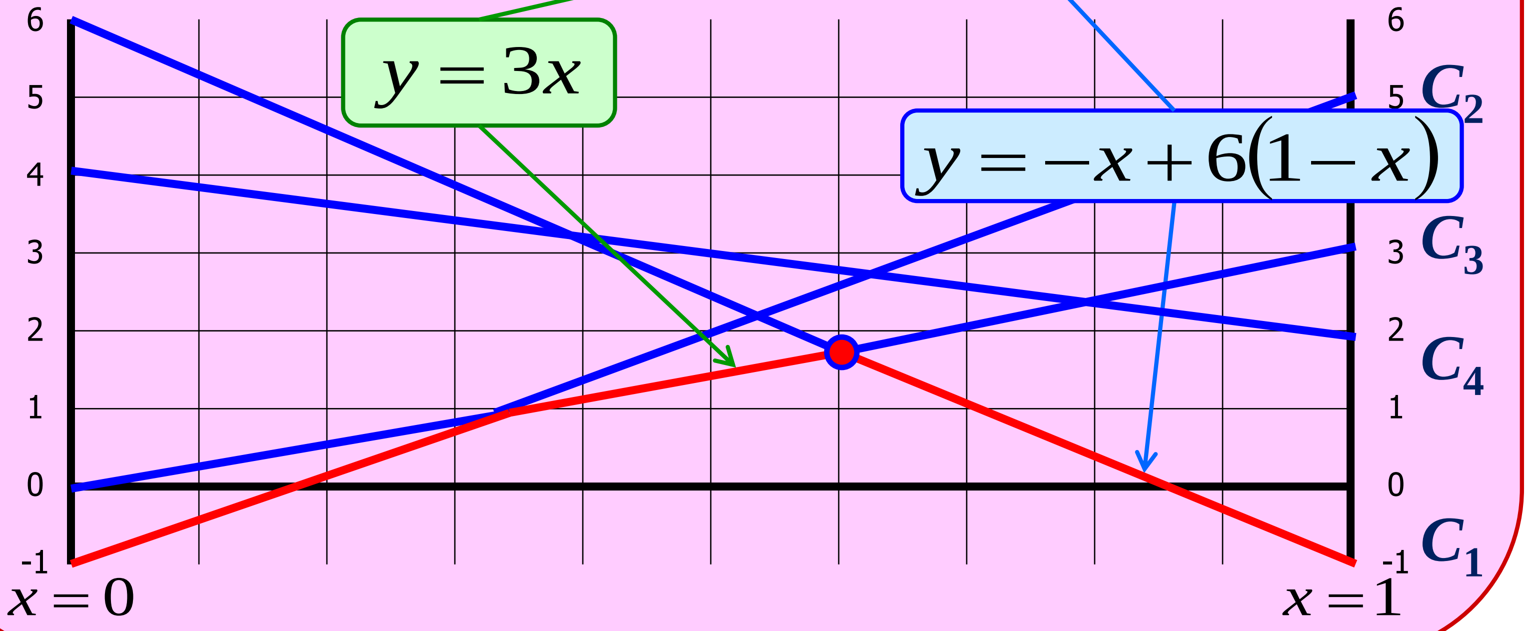
3. Find the value of *p* for the maximum point.



Example 1

$$A=\begin{pmatrix} \boxed{-1} & 5 & \boxed{3} & 2 \\ \boxed{6} & -1 & \boxed{0} & 4 \end{pmatrix}$$

3. Find the value of *p* for the maximum point.



$y=3x$

$y=-x+6(1-x)$

Example 1

The value of *p* and *v* can also be obtained by solving

$$\begin{cases} y = 3x \\ y = -x + 6(1-x) \end{cases} \Rightarrow \begin{aligned} 3x &= -x + 6(1-x) \\ 3x &= -x + 6 - 6x \\ 10x &= 6 \\ x &= 0.6 \\ y &= 3(0.6) = 1.8 \end{aligned}$$

Therefore

$\boxed{p = 0.6}$ **and** $\boxed{v = 1.8}$

Example 1

We may also reduce

$$A = \begin{pmatrix} \boxed{\begin{matrix} -1 \\ 6 \end{matrix}} & \begin{matrix} 5 \\ -1 \end{matrix} & \boxed{\begin{matrix} 3 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 4 \end{matrix} \end{pmatrix}$$

to

$$\begin{pmatrix} -1 & 3 \\ 6 & 0 \end{pmatrix}$$

Then use oddment method and obtain

$$\mathbf{p} = (0.6, 0.4)$$

$$\mathbf{q} = (0.3, 0.7)$$

Example 1

Don't forget that there are 4 strategies for II.

$$A = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

Therefore

maximin strategy for I: $\mathbf{p} = (0.6, 0.4)$

minimax strategy for II: $\mathbf{q} = (0.3, 0, 0.7, 0)$

value: $v = 1.8$

Example 2

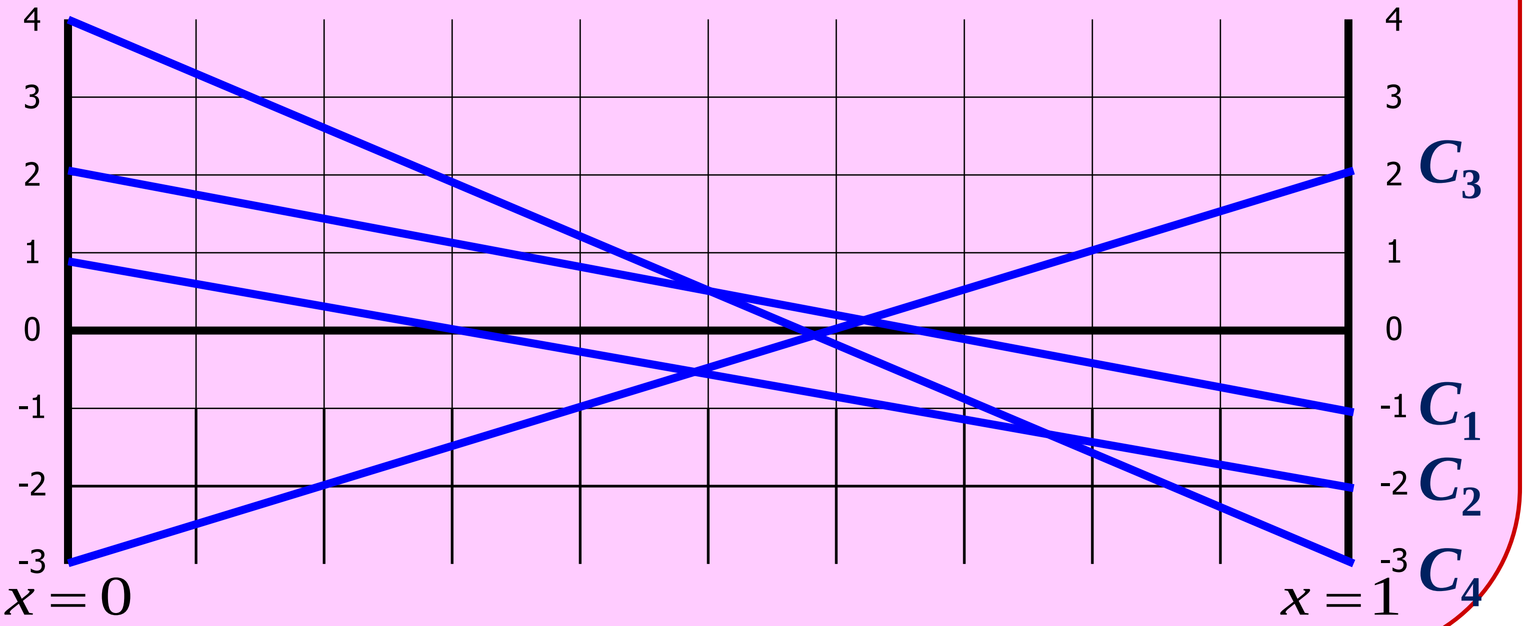
Solve the game matrix

$$A = \begin{pmatrix} -1 & -2 & 2 & -3 \\ 2 & 1 & -3 & 4 \end{pmatrix}$$

Example 2

$$A=\begin{pmatrix}-1 & -2 & 2 & -3\\ 2 & 1 & -3 & 4\end{pmatrix}$$

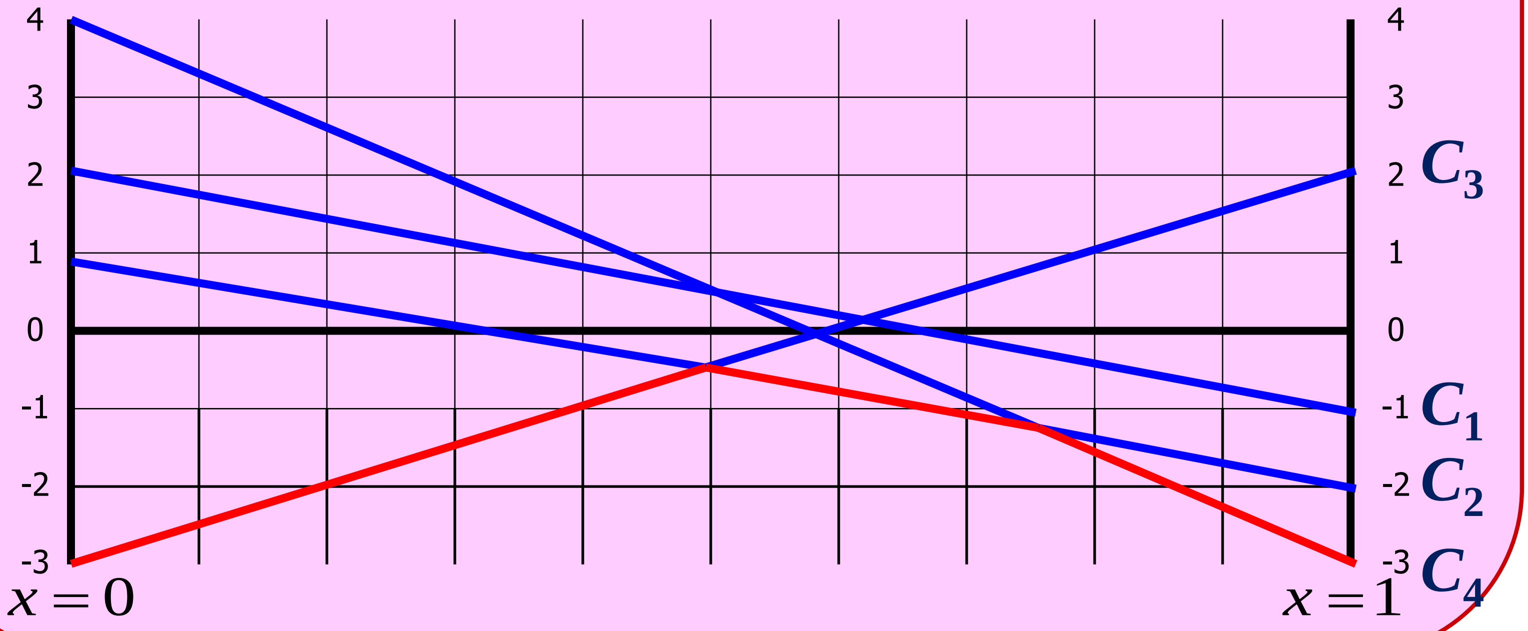
1. **Draw a line for each column.**



Example 2

$$A=\begin{pmatrix} -1 & -2 & 2 & -3 \\ 2 & 1 & -3 & 4 \end{pmatrix}$$

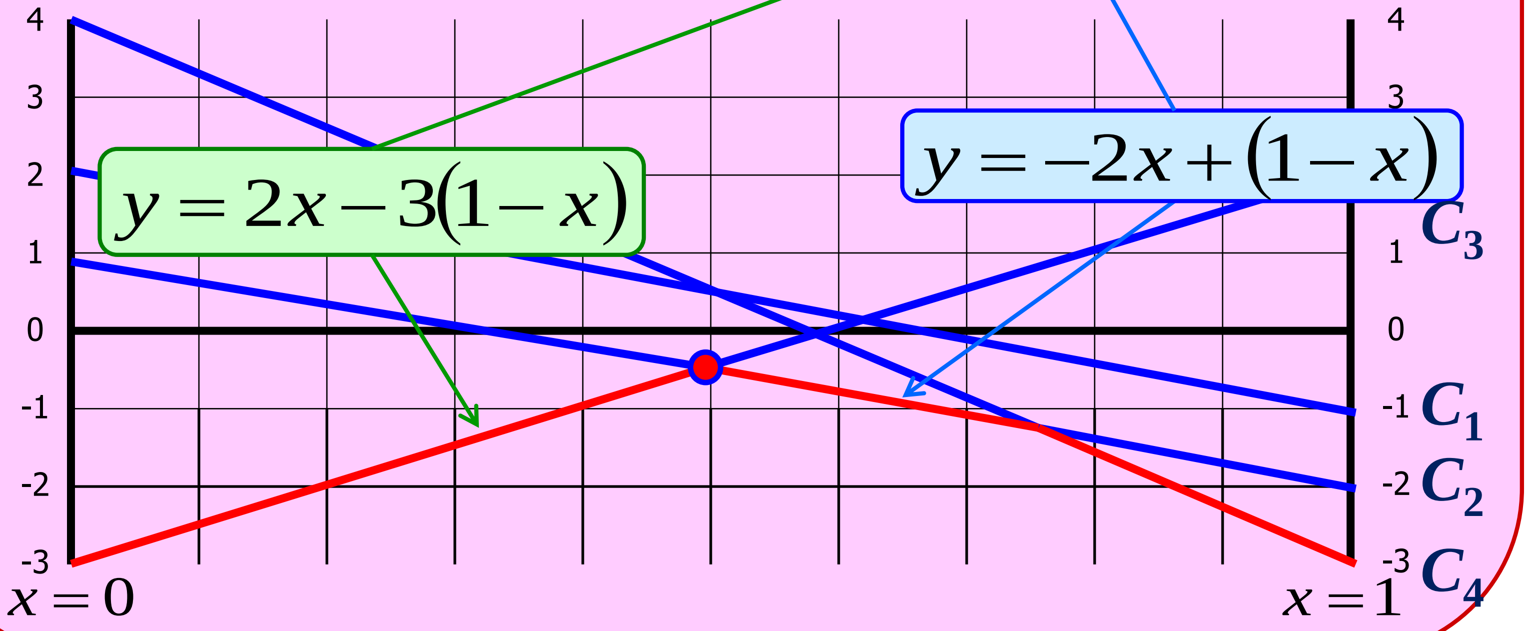
2. Draw the polygonal curve for minimum *y*.



Example 2

$$A = \begin{pmatrix} -1 & \boxed{\begin{matrix} -2 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 2 \\ -3 \end{matrix}} & -3 \\ 2 & & & 4 \end{pmatrix}$$

3. Find the value of *p* for the maximum point.



$y = 2x - 3(1 - x)$

$y = -2x + (1 - x)$

Example 2

We may reduce $A = \begin{pmatrix} -1 & \boxed{-2} & \boxed{2} & -3 \\ 2 & \boxed{1} & \boxed{-3} & 4 \end{pmatrix}$

to

$$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$$

Then use oddment method and obtain

$$\mathbf{p} = (0.5, 0.5)$$

$$\mathbf{q} = (0.625, 0.375)$$

Example 2

Therefore the solution of the game matrix

$$A = \begin{pmatrix} -1 & -2 & 2 & -3 \\ 2 & 1 & -3 & 4 \end{pmatrix}$$

is

maximin strategy for I: $\mathbf{p} = (0.5, 0.5)$

minimax strategy for II: $\mathbf{q} = (0, 0.625, 0.375, 0)$

value: $v = -0.5$

m-by-2 game

To solve *m*-by-2 game

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{pmatrix}$$

we may first solve the 2-by-*m* game

$$A' = -A^T = \begin{pmatrix} -a_{11} & -a_{21} & \cdots & -a_{m1} \\ -a_{12} & -a_{22} & \cdots & -a_{m2} \end{pmatrix}$$

m-by-2 game

and obtain

- **maximin strategy for *A'*:** $\mathbf{p}' = (p_1', p_2')$
- **minimax strategy for *A'*:** $\mathbf{q}' = (q_1', \cdots, q_m')$
- **value of *A'*:** v'

Then

- **maximin strategy for *A*:** $\mathbf{p} = \mathbf{q}' = (q_1', \cdots, q_m')$
- **minimax strategy for *A*:** $\mathbf{q} = \mathbf{p}' = (p_1', p_2')$
- **value of *A*:** $v = -v'$

Example

Solve the game matrix

$$A = \begin{pmatrix} 1 & -6 \\ -5 & 1 \\ -3 & 0 \\ -2 & -4 \end{pmatrix}$$

Example

Solving

$$A' = -A^T = \begin{pmatrix} -1 & 5 & 3 & 2 \\ 6 & -1 & 0 & 4 \end{pmatrix}$$

we have

maximin strategy of *A'*: $\mathbf{p}' = (0.6, 0.4)$

minimax strategy of *A'*: $\mathbf{q}' = (0.3, 0, 0.7, 0)$

value of *A'*: $v' = 1.8$

Example

Therefore the solution to

$$A = \begin{pmatrix} 1 & -6 \\ -5 & 1 \\ -3 & 0 \\ -2 & -4 \end{pmatrix}$$

is

maximin strategy of *A*: $\mathbf{p} = \mathbf{q}' = (0.3, 0, 0.7, 0)$

minimax strategy of *A*: $\mathbf{q} = \mathbf{p}' = (0.6, 0.4)$

value of *A*: $v = -v' = -1.8$

Dominated strategy

1. **We say that row R_1 dominates row R_2 if every entry of R_1 is larger than or equal to the corresponding entry of R_2.**
2. **We say that column C_1 dominates column C_2 if every entry of C_1 is less than or equal to the corresponding entry of C_2.**

If a row (column) is dominated by another row (column), then it can be removed when finding the solution of the game.

Example 1

Solve the game

$$A = \begin{pmatrix} 3 & 2 & -2 & 1 \\ -1 & -2 & 3 & 0 \\ 0 & -1 & 4 & 3 \end{pmatrix}$$

Example 1

The second row is dominated by the third row.

$$A=\begin{pmatrix} 3 & 2 & -2 & 1 \\ \xcancel{-1} & \xcancel{-2} & \xcancel{3} & \xcancel{0} \\ 0 & -1 & 4 & 3 \end{pmatrix}$$

The matrix is reduced to

$$A'=\begin{pmatrix} 3 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \end{pmatrix}$$

Example 1

Solving the 2-by-4 matrix

$$A' = \begin{pmatrix} 3 & 2 & -1 & 1 \\ 0 & -1 & 4 & 3 \end{pmatrix}$$

the solution to *A’* is

$$\mathbf{p}' = (0.8,0.2) \quad \mathbf{q}' = (0,0.4,0,0.6) \quad v' = 1.2$$

Don’t forget that I has three strategies.

Now we may write down the solution to *A*

$$\mathbf{p} = (0.8,0,0.2) \quad \mathbf{q} = (0,0.4,0,0.6) \quad v = 1.2$$

Example 2

Solve the game

$$A = \begin{pmatrix} 1 & -1 & -3 & 4 \\ -3 & -2 & 2 & 1 \\ 1 & -2 & -4 & -3 \\ 2 & 0 & -1 & 3 \end{pmatrix}$$

Example 2

1. 3rd row is dominated by 4th row

$$A = \begin{pmatrix} 1 & -1 & -3 & 4 \\ -3 & -2 & 2 & 1 \\ \cancel{1} & \cancel{-2} & \cancel{-4} & \cancel{-3} \\ 2 & 0 & -1 & 3 \end{pmatrix}$$

Example 2

2. 4th column is dominated by 2nd column.

$$A=\begin{pmatrix} 1 & -1 & -3 & \cancel{4} \\ -3 & -2 & 2 & \cancel{1} \\ \cancel{1} & \cancel{-2} & \cancel{-4} & \cancel{-3} \\ 2 & 0 & -1 & \cancel{3} \end{pmatrix}$$

Example 2

3. 1st row is dominated by 4th row.

$$A=\begin{pmatrix} \cancel{1} & \cancel{1} & \cancel{-3} & \cancel{4} \\ -3 & -2 & 2 & \cancel{1} \\ \cancel{1} & \cancel{-2} & \cancel{-4} & \cancel{-3} \\ 2 & 0 & -1 & \cancel{3} \end{pmatrix}$$

Example 2

The solution to the reduced matrix

$$A' = \begin{pmatrix} -3 & -2 & 2 \\ 2 & 0 & -1 \end{pmatrix}$$

is

$$\mathbf{p}' = (0.2, 0.8) \qquad \mathbf{q}' = (0, 0.6, 0.4) \qquad \mathbf{v}' = -0.4$$

Therefore the solution to *A* is

$$\mathbf{p} = (\boxed{0}, 0.2, \boxed{0}, 0.8) \qquad \mathbf{q} = (0, 0.6, 0.4, \boxed{0}) \qquad \mathbf{v} = -0.4$$

Add the deleted rows and columns back

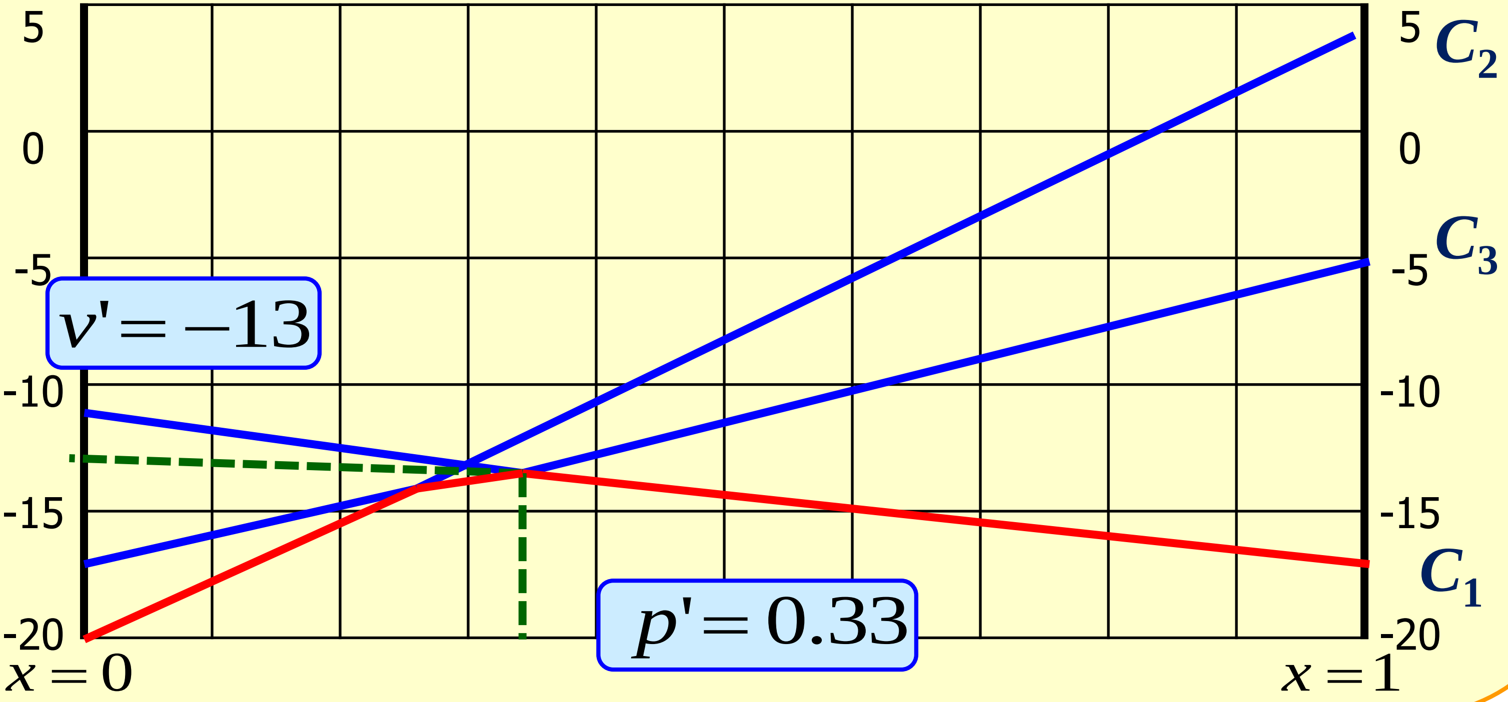
Exercise 1

Solve

$$A = \begin{pmatrix} 17 & 11 \\ -4 & 20 \\ 5 & 17 \end{pmatrix}$$

Exercise 1

$$A' = -A^T = \begin{pmatrix} -17 & 4 & -5 \\ -11 & -20 & -17 \end{pmatrix}$$



Exercise 1

$$A' = -A^T = \begin{pmatrix} -17 & 4 & -5 \\ -11 & -20 & -17 \end{pmatrix}$$

the solution to *A'* is

$\mathbf{p}' = (0.33, 0.67)$ $\mathbf{q}' = (0.67, 0, 0.33)$ $v' = -13$

The solution to *A* is

$\mathbf{p} = (0.67, 0, 0.33)$ $\mathbf{q} = (0.33, 0.67)$ $v = 13$

Jamaican fishing

William Davenport:
Jamaican fishing, a game theory analysis;
Yale University Publications in Anthropology, No. 59

Jamaican fishing

Average profit per month of fisherman

		Current	
		Run	**Not run**
Fisherman	**Inside only**	**17.3**	**11.5**
	Outside only	**-4.4**	**20.6**
	In and Out	**5.2**	**17.0**

Jamaican fishing

$$A = \begin{pmatrix} 17.3 & 11.5 \\ -4.4 & 20.6 \\ 5.2 & 17.0 \end{pmatrix}$$

	Minimax solution	**Real situation**
Fisherman	**(0.67,0,0.33)**	**(0.69,0,0.31)**
Current	**(0.31,0.69)**	**(0.25,0.75)**
Expected profit	**13.31**	**13.29**

Jamaican fishing

When current uses $\mathbf{q} = (0.25, 0.75)$,

$$A\mathbf{q} = \begin{pmatrix} 17.3 & 11.5 \\ -4.4 & 20.6 \\ 5.2 & 17.0 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 12.95 \\ 14.35 \\ 14.05 \end{pmatrix}$$

The best strategy for the fisherman is (0,1,0), i.e. always fishing outside. However it is a relatively risky strategy.

Exercise 2

Solve the game

$$A = \begin{pmatrix} 3 & 5 & 6 & 4 \\ 4 & 8 & 7 & 5 \\ 6 & 3 & 1 & 2 \\ 2 & 5 & 3 & 4 \end{pmatrix}$$

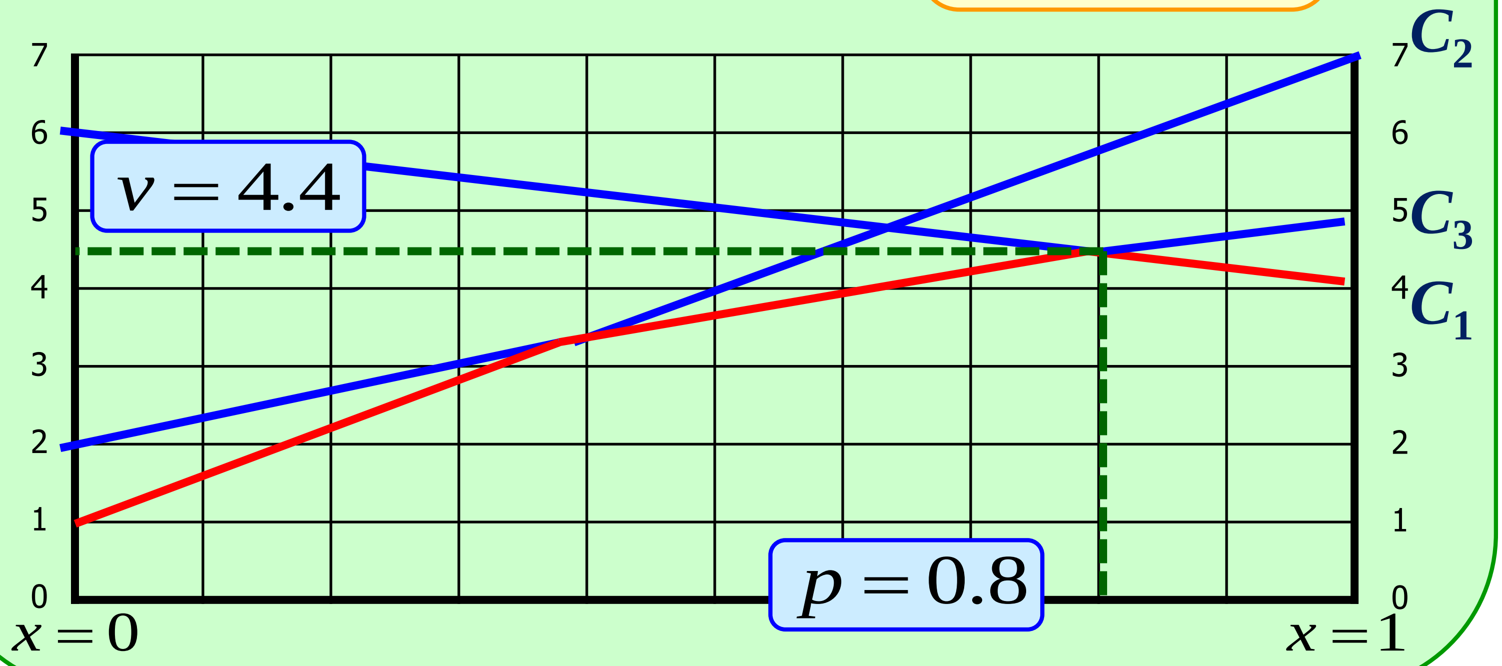
Exercise 2

Delete the dominated strategies in the order R_4, C_2, R_1.

$$A = \begin{pmatrix} \cancel{3} & \cancel{5} & \cancel{6} & \cancel{4} \\ 4 & \cancel{8} & 7 & 5 \\ 6 & \cancel{3} & 1 & 2 \\ \cancel{2} & \cancel{5} & \cancel{3} & \cancel{4} \end{pmatrix}$$

Exercise 2

The reduced matrix is $\begin{pmatrix} 4 & 7 & 5 \\ 6 & 1 & 2 \end{pmatrix}$



Exercise 2

Therefore the solution is

maximum strategy for I: $\boxed{\mathbf{p} = (0, 0.8, 0.2, 0)}$

minimax strategy for II: $\boxed{\mathbf{q} = (0.6, 0, 0, 0.4)}$

value of *A*: $\boxed{v = 4.4}$

Colonel Blotto game

Colonel Blotto was tasked to distribute his soldiers over 3 battlefields knowing that on each battlefield the party that has allocated the most soldiers will win and the payoff is the number of winning fields minus the number of losing fields.

Colonel Blotto game

If Colonel Blotto has *n* platoons, then the total number of strategies he has is $\boxed{C_2^{n+2}}$.

Example: When *n* = 4, Colonel Blotto has $\boxed{C_2^6 = 15}$ strategies.

Colonel Blotto game

Suppose Colonel Blotto has 4 platoons and his enemy has 3 platoons. Then Colonel Blotto has 15 strategies while his enemy has 10 strategies. The game is represented by a 15-by-10 matrix.

Colonel Blotto game

However, we may simply it to a 4-by-3 matrix.

		Enemy		
		300	210	111
Colonel Blotto	400	1/3	-1/3	-1/3
	310	2/3	1/3	0
	220	1/3	2/3	1
	211	1	2/3	1/3

Expected payoffs of Colonel Blotto

Colonel Blotto game

Expected payoff when Colonel Blotto uses 220 strategy and Enemy uses 300 strategy is calculated in the table. Both players have 3 ways to distribute their army, so there are 9 possibilities.

Blotto	**Enemy**	**Payoff**
220	300	-1+1+0=0
220	030	1+(-1)+0=0
220	003	1+1+(-1)=1
202	300	-1+0+1=0
202	030	1+(-1)+1=1
202	003	1+0+(-1)=0
022	300	-1+1+1=1
022	030	0+(-1)+1=0
022	003	0+1+(-1)=0
Expected payoff:		1/3

Colonel Blotto game

We may also fix the distribution of Blotto's army. To calculate the expected payoff when Colonel Blotto uses 310 and Enemy uses 210, only 6 distributions of Enemy's army are needed to be considered.

Blotto	**Enemy**	**Payoff**
310	210	1
310	201	1
310	120	0
310	102	1
310	021	-1
310	012	0
Expected payoff:		1/3

Colonel Blotto game

Then we need to solve the game matrix

$$\begin{pmatrix} 1/3 & -1/3 & -1/3 \\ 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 1 \\ 1 & 2/3 & 1/3 \end{pmatrix}$$

Colonel Blotto game

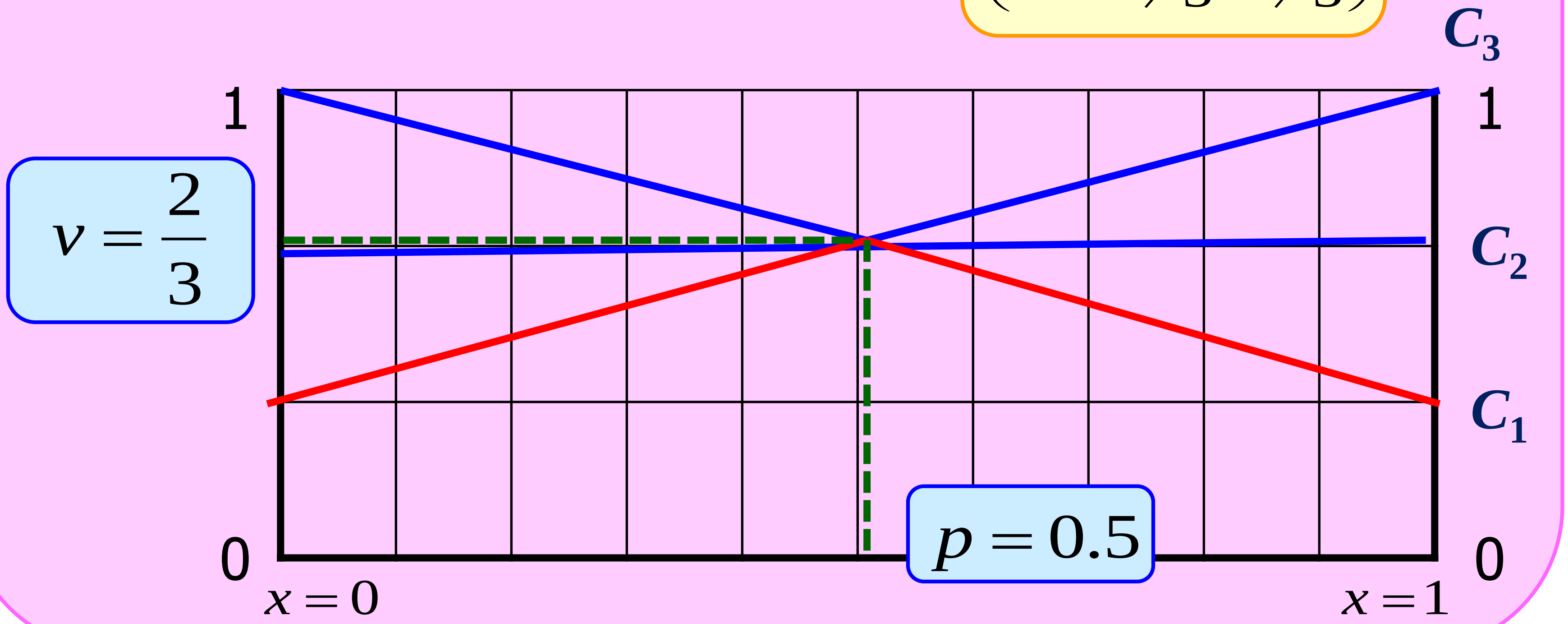
The first two rows are dominated.

$$\begin{pmatrix} \cancel{1/3} & \cancel{-1/3} & \cancel{-1/3} \\ \cancel{2/3} & \cancel{1/3} & \cancel{0} \\ 1/3 & 2/3 & 1 \\ 1 & 2/3 & 1/3 \end{pmatrix}$$

Colonel Blotto game

The matrix is reduced to

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$



$v=\frac{2}{3}$

$p=0.5$

Colonel Blotto game

maximum strategy for Colonel Blotto:

$$\mathbf{p} = (0,0,0.5,0.5)$$

(Using each of 220 and 211 with a probability of 0.5.)

minimax strategy for Enemy:

$$\mathbf{q} = (s,1-2s,s),\ \ 0 \le s \le 0.5$$

(For example using 210 constantly.)

value of the game:

$$v = \frac{2}{3}$$

Note that the minimax strategy for Enemy is not unique.